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HOMOGENEOUS SOLUTIONS AND SAINT-VENANT PROBLEMS FOR A NATURALLY TWISTED ROD†

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The method of homogeneous solutions is used to investigate a three-dimensional problem for a naturally twisted rod. A group of elementary solutions is established, enabling an applied theory of naturally twisted rods to be developed without involving any hypotheses, by rigorous mathematical methods, as done previously [1] for prismatic rods. It is shown that in the general case (arbitrary twist and arbitrary position of the cross-sectional centre of gravity relative to the screw axis) the construction of elementary solutions reduces to solving two types of boundary-value problem in the cross-section, which in turn reduces to variational problems for non-negative operators. A stiffness matrix is obtained which relates the components of the principal vector and principal momentum of the external stresses to the coefficients of expansions in series of elementary solutions (the latter may be considered as generalized displacements). The Saint-Venant principle is substantiated. Copyright © 1996 Elsevier Science Ltd.

In the first investigations of the Saint-Venant problems of the stretching, twisting and bending of naturally twisted rods [2–5], most attention was paid to stretching-twisting, on the assumption that the non-dimensional relative angle of twist if small.

Based on an a priori assumption concerning the structure of a Saint-Venant-type solution, the three-dimensional problem has been reduced [6] to a system of 18 equations, but most of the research has been done on the stretching–twisting problem only.

1. NOTATION AND FORMULATION OF THE BOUNDARY-VALUE PROBLEM

For brevity, we shall refer to a twisted rod as a "pseudo-cylinder". The concept of a pseudo-cylinder includes such practical objects as drill-bits, turbine vanes and cylindrical rods. The domain V occupied by a pseudo-cylinder is obtained by screw motion of a plane figure S along the axis x_3 of a fixed Cartesian system of coordinates x_k. As a parameter we take the relative angle of twist τ and consider it to be constant. As in [6], we introduce a co-moving system of coordinates ξ_k, in which the directions of the ξ_α axes ($\alpha = 1, 2$) are rigidly attached to S as it moves. The radius vector of an arbitrary point of V in the co-moving system is written as

$$\mathbf{R} = \xi_k \mathbf{e}_k; \quad \varphi = \tau\xi$$

$$\mathbf{e}_1 = \mathbf{i}_1 \cos\varphi + \mathbf{i}_2 \sin\varphi, \quad \mathbf{e}_2 = -\mathbf{i}_1 \sin\varphi + \mathbf{i}_2 \cos\varphi, \quad \mathbf{e}_3 = \mathbf{i}_3$$

where $\mathbf{i}_k$ are the unit vectors of the fixed system of coordinates and $\mathbf{e}_k$ are those of the co-moving one.

Other notation: Γ denotes the lateral surface, ∂S is the boundary of S, N_k is the projection of the vector of the outer normal to Γ, n_α is the projection of the external normal to ∂S onto the appropriate axis of the co-moving system of coordinates and $\xi^0{}_\alpha$ denote the coordinates of points of ∂S. Unless otherwise indicated, Latin subscripts take values from 1 to 3, Greek subscripts take values 1 and 2, and summation is assumed to be performed over repeated subscripts. The projections N_k and n_α satisfy the relationships

$$N_\alpha = cn_\alpha, \quad N_3 = \tau b$$

$$c = (1+\tau^2 b)^{-1/2}, \quad b = \xi_\alpha^0 \xi_\alpha^{0\prime}, \quad \xi_\alpha^{0\prime} = d\xi_\alpha^0 / ds$$

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Below we shall use the Hilbert spaces H_1 and H_2 of three-component complex-valued vector functions defined on S and ∂S, respectively, with inner products

$$(\mathbf{a}_1,\mathbf{a}_2)_1 = \int_S \mathbf{a}_1 \cdot \bar{\mathbf{a}}_2 dS = \int_S a_{k1}\bar{a}_{k2} dS$$

$$(\mathbf{a}_1,\mathbf{a}_2)_2 = \int_{\partial S} \mathbf{a}_1 \cdot \bar{\mathbf{a}}_2 ds = \int_{\partial S} a_{k1}\bar{a}_{k2} ds$$

as well as the space $H = H_1 \times H_2$ with inner product

$$(\mathbf{a}_1, \mathbf{a}_2) = (\mathbf{a}_1, \mathbf{a}_2)_1 + (\mathbf{a}_1, \mathbf{a}_2)_2$$

Throughout, $\mathbf{a}_k = \{a_{k1}, a_{k2}, a_{k3}\}$, and $\bar{a}_{kj}$ denotes complex conjugates.

We shall assume that the lateral surface Γ of the pseudo-cylinder is unstressed. The equilibrium equations and boundary conditions on Γ may be written in operator notation as follows:

$$\mathcal{L}(-i\partial)\mathbf{u} \equiv -\partial^2 C\mathbf{u} - i\partial B\mathbf{u} + A\mathbf{u} = \mu^{-1}\mathbf{K} \tag{1.1}$$

$$(E - i\partial G)\mathbf{u}|_\Gamma = 0 \tag{1.2}$$

where $\mathbf{u}$ is the vector of displacements, and A, B, C, E and G are matrix operators with the following elements

$$A_{11} = -2(1+\varkappa)\partial_1^2 - \partial_2^2 - \tau^2(D^2-1), \quad A_{12} = -(1+2\varkappa)\partial_1\partial_2 + 2\tau^2 D$$

$$A_{13} = -\tau(1+2\varkappa)\partial_1 D, \quad A_{21} = -(1+2\varkappa)\partial_1\partial_2 - 2\tau^2 D$$

$$A_{22} = -\partial_1^2 - 2(1+\varkappa)\partial_2^2 - \tau^2(D^2-1), \quad A_{23} = -\tau(1+2\varkappa)\partial_2 D$$

$$A_{31} = -\tau(1+2\varkappa)D\partial_1, \quad A_{32} = -\tau(1+2\varkappa)D\partial_2$$

$$A_{33} = -\partial_1^2 - \partial_2^2 - 2(1+\varkappa)\tau^2 D^2$$

$$B_{11} = B_{22} = -2i\tau D, \quad B_{12} = -B_{21} = 2i\tau$$

$$B_{13} = B_{31} = -i(1+2\varkappa)\partial_1, \quad B_{23} = B_{32} = -i(1+2\varkappa)\partial_2$$

$$B_{33} = -4i\tau(1+\varkappa)D$$

$$C_{11} = C_{22} = 1, \quad C_{33} = 2(1+\varkappa), \quad C_{kl} = 0, \quad k \neq l$$

$$E_{11} = 2(1+\varkappa)n_1\partial_1 + n_2\partial_2 + \tau^2 bD, \quad E_{12} = 2\varkappa n_1\partial_2 + n_2\partial_1 - \tau^2 b$$

$$E_{13} = 2\varkappa n_1\tau D + \tau b\partial_1, \quad E_{21} = n_1\partial_2 + 2\varkappa n_2\partial_1 + \tau^2 b$$

$$E_{22} = n_1\partial_1 + 2(1+\varkappa)n_2\partial_2 + \tau^2 bD, \quad E_{23} = 2\varkappa n_2\tau D + \tau b\partial_2$$

$$E_{31} = n_1\tau D + n_2\tau + 2\varkappa\tau b\partial_1, \quad E_{32} = -n_1\tau + n_2\tau D + 2\varkappa\tau b\partial_2$$

$$E_{33} = n_1\partial_1 + n_2\partial_2 + 2(1+\varkappa)\tau^2 bD$$

$$G_{11} = G_{22} = \tau b, \quad G_{12} = G_{21} = 0, \quad G_{13} = 2\varkappa n_1, \quad G_{23} = 2\varkappa n_2$$

$$G_{31} = n_1, \quad G_{32} = n_2, \quad G_{33} = (2+\varkappa)\tau b$$

$$\partial_\alpha = \partial/\partial\xi_\alpha, \quad \partial = \partial/\partial\xi, \quad D = \xi_2\partial_1 - \xi_1\partial_2, \quad \varkappa = \frac{\nu}{1-2\nu}$$

$\mathbf{K}$ denotes the vector of body forces, μ is the shear modulus and ν is Poisson's ratio.

For the final formulation of the boundary-value problem, we need conditions to be satisfied at $\xi = 0$ and $\xi = 1$. We will return to this later.

2. HOMOGENEOUS SOLUTIONS

Putting $K = 0$ in (1.1), we will seek a solution in the form

$$\mathbf{u} = e^{i\gamma\xi}\mathbf{a}(\xi_1, \xi_2) \tag{2.1}$$

Substituting (2.1) into (1.1), we obtain the spectral problem in the cross-section

$$L(\gamma)\mathbf{a} = \{\mathcal{L}(\gamma)\mathbf{a}, M(\gamma)\mathbf{a}|_{\partial S}\} = 0 \tag{2.2}$$

where ∂S is the boundary of S.

There is a voluminous literature on spectral problems of this type. A brief review, also touching on open problems, may be found in [7].

Recall that a vector function

$$\mathbf{u}_q(\xi) = e^{i\gamma_q\xi}\mathbf{a}_q \tag{2.3}$$

is called an elementary solution (ES) satisfying the homogeneous equation (1.1) and the boundary condition (1.2) if γ_q is a simple eigenvalue and $\mathbf{a}_q$ is a corresponding eigenvector.

If γ_k is a multiple eigenvalue, associated with a Jordan chain $\mathbf{a}_{0q}, \ldots, \mathbf{a}_{pq}$, where $\mathbf{a}_{0q}$ is an eigenvector and $\mathbf{a}_{lq}$ ($l = 1, 2, \ldots, p$) are associated vectors, then every Jordan chain may be associated with a whole group of elementary solutions

$$\mathbf{u}_{qn}(\xi) = \frac{(i\xi)^n}{n!}\mathbf{a}_{q0} + \frac{(i\xi)^{n-1}}{(n-1)!}\mathbf{a}_{q1} + \ldots + \mathbf{a}_{qn} \tag{2.4}$$

The total number of these ESs is equal to the algebraic multiplicity of the eigenvalue γ_k.

The associated vectors are determined by solving the following boundary-value problems

$$L(\gamma_q)\mathbf{a}_{qs} = \boldsymbol{\psi}_{qs} \quad \boldsymbol{\psi}_{qs} = \{\mathbf{F}_{qs}, \mathbf{f}_{qs}\} \tag{2.5}$$

$$\mathbf{F}_{q1} = -\mathcal{L}'(\gamma_q)\mathbf{a}_{q0}$$

$$\mathbf{F}_{qs} = -\mathcal{L}'(\gamma_q)\mathbf{a}_{qs-1} - \frac{1}{2}\mathcal{L}''(\gamma_q)\mathbf{a}_{qs-2}, \quad s = 2, 3, \ldots, p$$

$$\mathbf{f}_{qs} = -G\mathbf{a}_{qs}|_{\partial S}, \quad s = 1, 2, \ldots, p$$

Since each of problems (2.5) is a "problem on the spectrum", it follows that a necessary condition for them to be solvable is that

$$\int_S \mathbf{F}_{qs} \cdot \bar{\mathbf{a}}_{q0} dS + \int_{\partial S} \mathbf{f}_{qs} \cdot \bar{\mathbf{a}}_{q0} ds = 0 \tag{2.6}$$

It has been shown [1] that the classical Saint-Venant solutions [8] are linear combinations of 12 ESs corresponding to the eigenvalue $\gamma_0 = 0$. An analogous result may be established for pseudo-cylinders.

We first observe that

$$\overset{1}{\mathbf{a}}_0 = \{0, 0, 1\}, \quad \overset{2}{\mathbf{a}}_0 = \{-\xi_2, \xi_1, 0\}$$
$$\overset{3}{\mathbf{a}}_0 = \{1, i, 0\}, \quad \overset{4}{\mathbf{a}}_0 = \{1, -i, 0\} \tag{2.7}$$

are eigenvectors of the spectral problem (2.2). Here $\overset{1}{\mathbf{a}}_0$, $\overset{2}{\mathbf{a}}_0$ correspond to the eigenvalue $\gamma_0 = 0$, $\overset{3}{\mathbf{a}}_0$ corresponds to the eigenvalue $\gamma_1 = \tau$ and $\overset{4}{\mathbf{a}}_0$ corresponds to $\gamma_{-1} = -\tau$.

This result is obtained from the following representation for a vector defining the group of rigid displacements of the axial section S

$$u_1 = \frac{1}{2}(u_1^0 - iu_2^0)e^{i\tau\xi} + \frac{i}{2}(u_1^0 + iu_2^0)e^{-i\tau\xi} - \omega\xi_2$$

$$u_2 = \frac{1}{2}(u_2^0 - iu_1^0)e^{i\tau\xi} + \frac{i}{2}(u_1^0 + iu_2^0)e^{-i\tau\xi} + \omega\xi_1$$

$$u_3 = u_3^0$$

where u_j^0 are the components of a linear displacements and ω is a small rotation around the ξ axis.

We will show that to each eigenvector there corresponds an appropriate system of associated vectors.

We first construct a Jordan chain for the eigenvector $\mathbf{a}_j^0$ ($j = 1, 2$). Setting $\gamma_q = 0$, $\mathbf{a}_{qs} = \mathbf{a}_1^j$ in (2.5), we obtain the following boundary-value problems

$$A\mathbf{a}_1^j = 0, \quad G\mathbf{a}_1^j|_{\partial S} = \mathbf{f}_1^j \tag{2.8}$$

$$\mathbf{f}_1^1 = \{2\varkappa in_1, 2\varkappa in_2, 2i(1+\varkappa)\tau b\}, \quad \mathbf{f}_1^2 = \{\tau ib\xi_2, -\tau ib\xi_1, ib\}$$

It is easy to verify that the solvability conditions (2.6) are satisfied for these problems. The problems for $\mathbf{a}_1^2$, $\mathbf{a}_2^2$ are unsolvable, from which it follows that each of the eigenvectors $\mathbf{a}_0^1$, $\mathbf{a}_0^2$ has only one associated vector. The eigenvectors $\mathbf{a}_0^3$, $\mathbf{a}_0^4$ have three associated vectors each, and

$$\mathbf{a}_1^3 = \{0, 0, -i\zeta\}, \quad \mathbf{a}_1^4 = \{0, 0, -i\bar{\zeta}\} \tag{2.9}$$

$$\zeta = \xi_1 + i\xi_2$$

For $\mathbf{a}_2^3$, $\mathbf{a}_3^3$, $\mathbf{a}_2^4$, $\mathbf{a}_3^4$ we obtain boundary-value problems similar to (2.5), except that instead of γ_k we substitute τ and $-\tau$, respectively. Note that it will suffice to construct a Jordan chain for τ, because for $-\tau$

$$\mathbf{a}_j^4 = (-1)^{j-1}\bar{\mathbf{a}}_j^3 \tag{2.10}$$

The expressions for the vectors $\mathbf{F}_l^3$, $\mathbf{f}_l^3$ ($l = 2, 3$) are

$$\mathbf{F}_2^3 = 2\varkappa\mathbf{a}_0^3, \quad \mathbf{f}_2^3 = \{-2\varkappa n_1\zeta, -2\varkappa n_2\zeta, -2(1+\varkappa)\tau b\zeta\}$$

$$\mathbf{F}_{13}^3 = i[2\tau(D+i)a_{12}^3 - 2\tau a_{22}^3 + (1+2\varkappa)\partial_1 a_{32}^3]$$

$$\mathbf{F}_{23}^3 = i[2\tau a_{12}^3 + 2\tau(D+i)a_{22}^3 + (1+2\varkappa)\partial_2 a_{32}^3]$$

$$\mathbf{F}_{33}^3 = i[(1+2\varkappa)(\partial_1 a_{12}^3 + \partial_2 a_{23}^3) + 4\tau(1+\varkappa)(D+i)a_{32}^3 + 2(1+\varkappa)\zeta]$$

$$\mathbf{f}_{13}^3 = -i[2\varkappa n_1 a_{32}^3 + \tau b a_{12}^3]$$

$$\mathbf{f}_{23}^3 = -i[2\varkappa n_2 a_{32}^3 + \tau b a_{22}^3]$$

$$\mathbf{f}_{33}^3 = -i[n_1 a_{12}^3 + n_2 a_{22}^3 + 2\tau(1+\varkappa)b a_{32}^3]$$

It is quite easy to prove the validity of the solvability conditions for $\mathbf{a}_2^3$, but somewhat more difficult to do so for $\mathbf{a}_3^3$.

3. VARIATIONAL FORMULATION

Consider the two boundary-value problems

$$A\mathbf{a} = 0, \quad G\mathbf{a}|_{\partial S} = \mathbf{f} \tag{3.1}$$

$$L(\pm\tau)\mathbf{a}^{\pm} = \mathbf{F}^{\pm}, \quad M(\pm\tau)\mathbf{a}^{\pm}|_{\partial S} = \mathbf{g}^{\pm} \tag{3.2}$$

Problem (3.1) symbolizes problems (2.8) while problem (3.2) symbolizes the boundary-value problems for $\mathbf{a}_l^j$ ($j = 3, 4$; $l = 2, 3$).

The boundary-value problem (3.2) generates in H an unbounded self-adjoint operator A with twodimensional kernel defined by the vectors $\mathbf{a}_0^1$, $\mathbf{a}_0^2$, problem (3.2) generates self-adjoint operators $L(\pm\tau)$ with one-dimensional kernels defined by the vectors $\mathbf{a}_0^3$, $\mathbf{a}_0^4$ respectively.

We shall assume that the solvability conditions (2.6) are satisfied for problems (3.1) and (3.2). Then the solutions of these boundary-value problems may be written as follows [9]:

$$\mathbf{a} = \mathbf{a}_* + C_1\mathbf{a}_0^1 + C_2\mathbf{a}_0^2 \tag{3.3}$$

$$\mathbf{a}^\pm = \mathbf{a}_*^\pm + C^\pm\mathbf{a}_0^\pm, \quad \mathbf{a}_0^+ = \mathbf{a}_0^3, \quad \mathbf{a}_0^- = \mathbf{a}_0^4$$

where C_1, C_2, $C^\pm$ are arbitrary constants.

The vector $\mathbf{a}_*$ may be determined y solving a variational problem for a quadratic functional

$$\Phi(\mathbf{a}) = \Phi_0(\mathbf{a}) - 2\int_{\partial S} f_k\bar{a}_k ds \tag{3.4}$$

$$\begin{aligned}\Phi_0(\mathbf{a}) = \int_S [2\varkappa|\partial_\alpha a_\alpha + \tau Da_3|^2 + |\partial_1 a_3 + \tau(Da_1 - a_2)|^2 + |\partial_2 a_3 + \tau(a_1 + Da_2)|^2 + \\ +2(|\partial_1 a_1|^2 + |\partial_2 a_2|^2) + |\partial_1 a_2 + \partial_2 a_1|^2 + 2\tau^2|Da_3|]dS\end{aligned} \tag{3.5}$$

In the subspace H_* whose elements satisfy the conditions

$$(\mathbf{a}, \mathbf{a}_0^j)_1 = 0, \quad j = 1, 2$$

the vector $\mathbf{a}_*$ is uniquely defined.

The vectors $\mathbf{a}_*^+$, $\mathbf{a}_*^-$ may be determined by solving variational problems for the functionals

$$\Phi^\pm(\mathbf{a}^\pm) = \Phi_0^\pm(\mathbf{a}^\pm) - 2Re\int_S F_k^\pm\bar{a}_k^\pm dS - 2Re\int_{\partial S} f_k^\pm\bar{a}_k^\pm ds \tag{3.6}$$

The expressions for the functionals $\Phi_0^\pm$ are obtained when D is replaced by $D \pm i$ in formula (3.5) and are positive-definite in the subspace $H^\pm$ whose elements satisfy the conditions

$$(\mathbf{a}, \mathbf{a}_0^\pm)_1 = 0$$

Thus, taking property (2.9) into account, we have reduced the construction of Saint-Venant's elementary solutions to two types of variational problem.

4. THE SAINT-VENANT ELEMENTARY SOLUTIONS

We define the Saint-Venant ESs as the subset of ESs corresponding to eigenvalues $\gamma_0 = 0, \gamma_{\pm 1} = \pm\tau$. It follows from the preceding analysis that this subset consists of 12 ESs, which we may write as follows:

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{a}_0^1, \quad \mathbf{u}_2 = \mathbf{a}_0^2, \quad \mathbf{u}_3(\xi) = \bar{\mathbf{u}}_4(\xi) = e^{i\tau\xi}\mathbf{a}_0^3 \\
&\mathbf{u}_6(\xi) = -\bar{\mathbf{u}}_5(\xi) = e^{-i\tau\xi}(i\xi\mathbf{a}_0^4 + \mathbf{a}_1^4) \\
&\boldsymbol{\sigma}_r = 0, \quad r = 1, 2, \ldots, 6 \\
&\mathbf{u}_7 = i\xi\mathbf{a}_0^1 + \mathbf{a}_1^1, \quad \mathbf{u}_8 = i\xi\mathbf{a}_0^2 + \mathbf{a}_2^2 \\
&\mathbf{u}_{10} = -\bar{\mathbf{u}}_9 = e^{-i\tau\xi}\left(-\frac{i\xi^3}{6}\mathbf{a}_0^4 - \frac{\xi^2}{2}\mathbf{a}_1^4 + i\xi\mathbf{a}_2^4 + \mathbf{a}_3^4\right) \\
&\mathbf{u}_{12} = -\bar{\mathbf{u}}_{11} = e^{-i\tau\xi}\left(-\frac{\xi^2}{2}\mathbf{a}_0^4 + i\xi\mathbf{a}_1^4 + \mathbf{a}_2^4\right) \\
&\boldsymbol{\sigma}_7 = \mathbf{b}_1^1, \quad \boldsymbol{\sigma}_8 = \mathbf{b}_1^2, \quad \boldsymbol{\sigma}_{10} = -\bar{\boldsymbol{\sigma}}_9 = e^{-i\tau\xi}(i\xi\mathbf{b}_2^4 + \mathbf{b}_3^4) \\
&\boldsymbol{\sigma}_{12} = \bar{\boldsymbol{\sigma}}_{11} = e^{-i\tau\xi}\mathbf{b}_2^4
\end{aligned} \tag{4.1}$$

where $\mu\sigma_1$ is the stress vector at the points of the cross-section S, and the vectors $\mathbf{b}_t^j$ are defined by

$$b_{\alpha t}^j = (\tau D + i\gamma)a_{1t}^j + (-1)^\alpha \tau a_{2t}^j + \partial_\alpha a_{3t}^j + i a_{\alpha t-1}^j$$

$$b_{3t}^j = 2\varkappa d_\alpha a_{\alpha t} + 2(1+\varkappa)(\tau D + ij)a_{3t}^j + 2i(1+\varkappa)a_{3t-1}^j$$

with $\gamma = 0$ for $j = 1, 2$, $\gamma = \tau$ for $j = 3$, and $\gamma = -\tau$ for $j = 4$.

It has been shown [1] how such a system can be used to construct a solution of the classical Saint-Venant problems. The method of construction carries over almost unchanged to the case of a pseudo-cylinder.

Suppose that the following conditions are stipulated at the ends of the cylinder

$$\mathbf{u}(0) = 0, \quad \boldsymbol{\sigma}(l) = \boldsymbol{\sigma}_* \tag{4.2}$$

A Saint-Venant solution of this boundary-value problem will be sought in the form

$$\mathbf{u}_l = \sum_{q=1}^{6} C_q \mathbf{u}_q(\xi) + \sum_{q=7}^{12} C_q \mathbf{u}_q(\xi - l) \tag{4.3}$$

where

$$\boldsymbol{\sigma}_l = \sum_{q=7}^{12} C_q \boldsymbol{\sigma}_q(\xi - l)$$

We introduce the notation

$$\mathbf{u}_q(0) = \mathbf{a}_q, \quad \boldsymbol{\sigma}_{6+q}(0) = \mathbf{b}_q, \quad q = 1, 2, \ldots, 6$$

By (4.1)

$$\mathbf{a}_j = \mathbf{a}_0^j \quad (j = 1, 2, 3, 4), \quad \mathbf{a}_5 = \mathbf{a}_1^3, \quad \mathbf{a}_6 = \mathbf{a}_1^4 \tag{4.4}$$

$$\mathbf{b}_1 = \mathbf{b}_1^1, \quad \mathbf{b}_2 = \mathbf{b}_1^2, \quad \mathbf{b}_3 = \mathbf{b}_3^3, \quad \mathbf{b}_4 = \mathbf{b}_3^4, \quad \mathbf{b}_5 = \mathbf{b}_2^3, \quad \mathbf{b}_6 = \mathbf{b}_2^4$$

Using the well-known properties of biorthogonality [9], it can be shown that among the distinct inner products

$$d_{qp} = (\mathbf{b}_q, \mathbf{a}_p)_1$$

d_{qq}, d_{12} and d_{21} do not vanish, and moreover

$$d_{11} = \int_S b_{31}^1 dS, \quad d_{12} = d_{21} = \int_S b_{31}^2 dS$$

$$d_{22} = \int_S (\xi_1 b_2^2 - \xi_2 b_1^2) dS$$

$$d_{33} = \ldots = d_{66} = d_* = i \int_S \bar{\zeta} b_{32}^3 dS$$

Substituting (4.3) into the second boundary condition (4.2), after successively multiplying by a_q and integrating, we obtain the following equations for $C_7, \ldots, C_{12}$

$$d_{11}C_7 + d_{12}C_8 = \mu^{-1}Q_3, \quad d_{12}C_7 + d_{22}C_8 = \mu^{-1}M_3 \tag{4.5}$$

$$C_9 = \bar{C}_{10} = \mu^{-1}d_*^{-1}(Q_1 - iQ_2), \quad C_{11} = -\bar{C}_{12} = \mu^{-1}d_*^{-1}i(M_1 + iM_2)$$

where Q_k and M_k are the components of the principal vector and principal moment of the external stresses σ_*. We emphasize that the components M_k are evaluated relative to the axes of the co-moving system of coordinates with $\xi = l$.

The constants C_q ($q = 1, 2, \ldots, 6$) may be determined as follows. Substituting (4.3) into the first boundary condition (4.2), after multiplying by $\mathbf{b}_q$ and integrating, we obtain the algebraic system of equations

$$\sum_{p=1}^{6} d_{qp} C_p = z_q \quad \left(z_q = -\left(\sum_{p=1}^{6} C_{6+p} \mathbf{u}_{6+p}(-l), \mathbf{b}_q \right)_1 \right) \tag{4.6}$$

In this method for constructing solutions of the Saint-Venant problems, the constants $C_7, \ldots, C_{12}$ are determined exactly (independently of the boundary layer), as in the case of the prismatic rod, which partially justifies the Saint-Venant principle for pseudo-cylinders (for a complete justification one must prove—see [10]—that the eigenvalue problem (2.2) has no real eigenvalues other than 0, τ and $-\tau$).

The exact values of the constants $C_1, \ldots, C_6$ in the general case cannot be determined without constructing a boundary layer, which is defined by the ESs belonging to the complex eigenvalues of problem (2.2). For small values of the parameter $\varepsilon = dl^{-1}$, where d is a certain characteristic linear magnitude associated with S, one can establish the asymptotic nature of the result. However, a proof of this fact and an analysis of specific problems are beyond the scope of this paper.

5. SUBSTANTIATION OF THE SAINT-VENANT PRINCIPLE FOR PSEUDO-CYLINDERS

As follows [7] from the fact that the elementary solutions of the homogeneous problem (1.1), (1.2) form a complete system, the stress vector in the section corresponding to solution of the three-dimensional problem may be expressed as

$$\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}_c + \sum_k \left[C_k^+ \boldsymbol{\sigma}_k^+(\xi) + C_k^- \boldsymbol{\sigma}_k^-(\xi - l) \right] \\ &\boldsymbol{\sigma}_k(\xi) = \mathbf{b}_k \exp(i\gamma_k \xi) \end{aligned} \tag{5.1}$$

Based on (5.1), justification of the Saint-Venant principle reduces to proving two propositions: (1) the principal vectors and principal moments $\boldsymbol{\sigma}_k^+$, $\boldsymbol{\sigma}_k^-$ vanish, (2) the inequalities $\operatorname{Im} \gamma_k^+ > 0$, $\operatorname{Im} \gamma_k^- < 0$ hold strictly. This is the scheme of the proof for an ordinary cylinder, as implemented in [10].

To prove the first proposition, consider the pair of vectors

$$\mathbf{V}_q = \{\mathbf{a}_q, \mathbf{b}_q\}, \quad \mathbf{V}_k^{\pm} = \{\mathbf{a}_k^{\pm}, \mathbf{b}_k^{\pm}\}$$

By the properties of biorthogonality [9], since $\mathbf{b}_q = 0$, we have

$$[\mathbf{V}_k^{\pm}, \mathbf{V}_q] = i[(\mathbf{a}_k^{\pm}, \mathbf{b}_q)_1 - (\mathbf{b}_k^{\pm}, \mathbf{a}_q)_1] = -i(\mathbf{b}_k^{\pm}, \mathbf{a}_q) = 0$$

Taking the specific forms of the vectors (4.4), (2.7) and (2.9) into account, we obtain the proof of the first proposition.

To prove the second proposition, multiply Eq. (2.2) by $\bar{\mathbf{a}}$. After integration and some elementary algebra, we obtain a quadratic equation

$$g_0 \gamma^2 + 2 g_1 \gamma + g_2 = 0 \tag{5.2}$$

where

$$\begin{aligned} &g_0 = c_1^2 + 2c_2^2 + 2c_3^2 \\ &g_1 = \operatorname{Im}[2(\mathring{\nabla}(\bullet)\mathbf{a}_0 + \tau D\mathbf{a}_3, \mathbf{a}_3)_\varkappa + (\mathring{\nabla}\mathbf{a}_3 + \tau\mathbf{a}_*, \mathbf{a}_0)_1 + 2\tau(D\mathbf{a}_3, \mathbf{a}_3)_1] \\ &g_2 = 2f_1^2 + f_1^2 + 2f_3^2 + 2n_1^2 + n_2^2 \\ &c_1 = \|\mathbf{a}_0\|_1, \quad c_2 = \|\mathbf{a}_3\|_1, \quad c_3 = \|\mathbf{a}_3\|_\varkappa \\ &f_1 = \|\mathring{\nabla}(\bullet)\mathbf{a}_0 + \tau D\mathbf{a}_3\|_\varkappa, \quad f_2 = \|\mathring{\nabla}\mathbf{a}_3 + \tau\mathbf{a}_*\|_1, \quad f_3 = \tau\|D\mathbf{a}_3\|_1 \\ &n_1^2 = \|\partial_1 a_1\|_1^2 + \|\partial_2 a_2\|_1^2, \quad n_2^2 = \|\partial_1 a_2 + \partial_2 a_1\|_1^2 \\ &\mathbf{a}_0 = a_\alpha \mathbf{e}_\alpha, \quad \mathbf{a}_* = (Da_1 - a_2)\mathbf{e}_1 + (Da_2 + a_1)\mathbf{e}_2 \\ &(\mathbf{a}, \mathbf{b})_\varkappa = \int_S \varkappa \mathbf{a} \cdot \bar{\mathbf{b}} \, dS, \quad \mathring{\nabla} = \mathbf{e}_\alpha \partial_\alpha \end{aligned}$$

Using the Cauchy–Bunyakovskii inequality, it can be proved that the discriminant of Eq. (5.2) satisfies the inequality $\Delta = g^2_1 - g_0 g_2 \leqslant 0$. It can also be shown that $\Delta = 0$ only for the eigenvector $\mathbf{a}_0^1$ and for linear combinations of $\mathbf{a}_0^2$, $\mathbf{a}_0^3$, $\mathbf{a}_0^4$. It follows that, apart from $\gamma_0 = 0$, $\gamma_1 = \tau$, $\gamma_{-1} = -\tau$, there are no other real eigenvalues, and so it follows from (5.1) that

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_c = O(\exp(-\alpha_1\theta))$$

$$(\alpha_1 = \operatorname{Im}\gamma_1^+ = -\operatorname{Im}\gamma_1^-, \quad \theta = \min(\xi, l - \xi))$$

which establishes the validity of the Saint-Venant principle.

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